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Everything about cubic function without calculus

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ABSTRACT. The purpose of this article is to show that all basic properties of cubic function that usually obtained with calculus (limits, derivatives) can be also obtained only by elementary algebra means.

MAIN RESULTS

1. Existence of a roots of reduced cubic polynomial. First we will prove that reduced cubic polynomial of the third degree

$$P(x) = x^3 + px + q$$

always have a root, i.e. solution of equation $P(x) = 0$.
By default everywhere further we assume that $p \neq 0$.

Lemma 1. (Preserve sign lemma). If $P(a) \neq 0$ then there is real $\varepsilon > 0$ such that for any $x \in (a - \varepsilon, a + \varepsilon)$ holds

$$signP(x) = signP(a).$$

Proof. Suppose that $|x - a| \leq 1$, then since

$$P(x) - P(a) = (x - a)\left(x^2 + xa + a^2 + p\right) =$$

$$= (x - a)\left((x - a)^2 + 3a(x - a) + 3a^2 + p\right) =$$

$$= (x - a)((x - a)^2 + 3a(x - a) + 3a^2 + p)$$

we have

$$|P(x) - P(a)| \leq |x - a|\left(1 + |p| + 3|a| + 3a^2\right).$$

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Using this inequality and taking

$$\varepsilon = \min\left\{1, \frac{|P(a)|}{2(1+|p|+3|a|+3a^2)}\right\}$$

we obtain that

$$|x-a| < \frac{|P(a)|}{2(1+|p|+3|a|+3a^2)}$$

for any $x \in (a-\varepsilon, a+\varepsilon)$ and, therefore,

$$|P(x) - P(a)| \leq \frac{|P(a)|}{2} \iff -\frac{|P(a)|}{2} + P(a) < P(x) < \frac{|P(a)|}{2} + P(a).$$

If $P(a) > 0$ then

$$P(x) > \frac{P(a)}{2} > 0;$$

If $P(a) < 0$ then

$$P(x) < \frac{|P(a)|}{2} + P(a) = \frac{P(a)}{2} < 0.$$

Lemma 2. If for some $a < b$ holds inequalities $P(a) < 0$ and $P(b) > 0$ then exist $x_* \in (a,b)$ such that $P(x_*) = 0$.

Proof. Let $\mathcal{N} := \{x \mid x \in (a,b) \text{ and } P(x) \leq 0\}$ .Since $\mathcal{N}$ bounded set then by Axiom of Completeness $\mathcal{N}$ have supremum- $\sup \mathcal{N}$ .Let $x_* := \sup \mathcal{N}$. It is mean that

1. For any $x \in \mathcal{N}$ holds $x \leq x_*$;
2. For any $\varepsilon > 0$ there is $x \in \mathcal{N}$ such that $x_* - \varepsilon < x$.

First we will prove that $P(x_*) \geq 0$. Suppose opposite, i.e. $P(x_*) < 0$.Then by Lemma 1 we can find some small enough positive real number h such that $P(x_* + h) < 0$. This mean that $x_* + h \in \mathcal{N}$ and, therefore, x_* isn't upper bound for $\mathcal{N}$, i.e. we get contradiction.
But if we suppose that $P(x_*) > 0$ then again by Lemma 1 there is real $\varepsilon > 0$ such that for any $x \in (x_* - \varepsilon, x_* + \varepsilon)$ holds $P(x) > 0$ and, in particular, $P(x) > 0$ for any $x \in (x_* - \varepsilon, x_*]$. This is the contradiction with property 2. for x_*.

Thus, remains only $P(x_*) = 0$.

Lemma 3. Cubic equation $P(x) = 0$ always have at least one real root.

Proof. Let $a := -\sqrt{|p| + |q| + 1}$ then $a < -1$ and

$$P(a) = a(a^2 + p) + q =$$

$$= a(|p| + p + |q| + 1) + q < -(|p| + p + |q| + 1) + q =$$

$$= -(|p| + p) - (|q| - q) - 1 \le -1 < 0.$$

Let $b = \sqrt{|p| + |q| + 1}$ then $b > 1$ and

$$P(b) = b(b^2 + p) + q =$$

$$= b(|p| + p + |q| + 1) + q > (|p| + p) + (|q| + q) + 1 \ge 1 > 0.$$

Hence, by Lemma 2 we have a root in (a, b).

2. Translation. Let $F(x) = x^3 - ax^2 + bx - c$ is monic polynomial of 3-rd degree and let $x = u + h$,then

$$F(u + h) = u^3 + 3u^2h + 3uh^2 + h^3 - au^2 - 2auh - aa^2 + bu +$$

$$+ bh - c = u^3 + u^2(3h - a) + u(3h^2 - 2ah + b) + F(h).$$

Denoting $F'(x) := 3x^2 - 2ax + b$ and $F''(x) := 6x - 2a$ we obtain

(1).

$$F(u + h) = u^3 + \frac{F''(h)}{2}u^2 + F'(h)u + F(h).$$

Using translation we always can reduce $F(x)$ to the cubic polynomial without second degree term. Namely, from claim

$$F''(h) = 0 \iff 3h - a = 0$$

follows desirable translation $x = u + \frac{a}{3}$. Thus and we obtain

$$F'\left(\frac{a}{3}\right)=p-\frac{a^2}{3}, F\left(\frac{a}{3}\right)=-\frac{2a^3}{27}+\frac{ab}{3}-c$$

and, therefore,

(2)

$$F\left(u+\frac{a}{3}\right)=u^3+pu+q,$$

where $p=b-\dfrac{a^2}{3}, q=\dfrac{ab}{3}-\dfrac{2a^3}{27}-c$ (**Translation Formulas**).
Using form (2) and Lemma 3 we immediately can conclude that any polynomial 3-d degree have a real root.

3. Wieta's conditions. Let x_1 is the root of $F(x)$ then

$$F(x)=F(x)-F(x_1)=(x-x_1)\left(x^2-x(a-x_1)+x_1^2+ax_1+b\right).$$

In supposition that quotient (quadratic trinomial)

$$Q(x)=x^2-x(a-x_1)+x_1^2+ax_1+b$$

have roots x_2 and x_3 then

$$Q(x)=(x-x_2)(x-x_3).$$

Hence,

$$F(x)=(x-x_1)(x-x_2)(x-x_3)=$$

$$=x^3-(x_1+x_2+x_3)x^2+(x_1x_2+x_2x_3+x_3x_1)x-x_1x_2x_3 \iff$$

(3). $$\begin{cases} x_1+x_2+x_3=a \\ x_1x_2+x_2x_3+x_3x_1=b \\ x_1x_2x_3=c \end{cases}.$$

From the other hand if x_1, x_2, x_3 satisfies to (3) then

$$F(x)=x^3-(x_1+x_2+x_3)x^2+(x_1x_2+x_2x_3+x_3x_1)x-x_1x_2x_3=$$

$$=(x-x_1)(x-x_2)(x-x_3)$$

and, therefore, x_1, x_2, x_3 are the roots of $F(x)$.

$$F(x) = x^3 - x_1x^2 - x_2x^2 - x_3x^2 + x_1x_2x + x_2x_3x + x_3x_1x - x_1x_2x_3 =$$

$$= x^2(x - x_1) - x_2x(x - x_1) - x_3(x - x_1) + x_2x_3(x - x_1) =$$

$$= (x - x_1)\left(x^2 - x_2x - x + x_2x_3\right) = (x - x_1)(x - x_2)(x - x_3).$$

Thus we obtain Vieta's Theorem for cubic equation.

Theorem. Cubic equation $x^3 - ax^2 + bx - c = 0$ have three real solution x_1, x_2, x_3 iff these numbers satisfy (3).

4. Further transformations and solution of cubic equation.
Substitution $u = kz$, where $k > 0$ in $u^3 + pu + q = 0$ gives us

$$k^3z^3 + pkz + q = 0 \iff \frac{k^2}{|p|}z^3 + \frac{p}{|p|}z + \frac{q}{|p|\,k} = 0$$

and we claim

$$\frac{k^2}{|p|} = \frac{4}{3} \iff$$

$$k = 2\sqrt{\frac{|p|}{3}}.$$

For such k we get equation

$$\frac{4}{3}z^3 + \frac{p}{|p|}z = -\frac{q\sqrt{3}}{2\,|p|\,\sqrt{|p|}} \iff$$

(4). $4z^3 + 3sign\,(p)\,z = d$, where $d := -\dfrac{3q\sqrt{3}}{2\,|p|\,\sqrt{|p|}}$.

Thus, we obtain that by appropriate linear transformation any cubic equation can be equivalently reduced to one of the two special cases- to $4z^3 - 3z = d$ if $p < 0$ or to $4t^3 + 3t = d$ if $p > 0$. Consider now the following cases:

1. $4z^3 + 3z = d$. Since $4z^3 + 3z = d$ have a solution and function $z \mapsto 4z^3 + 3z$ increasing on $\mathbb{R}$

then this solution is unique. We will find it, using substitution

$$z = \frac{1}{2}\left(t - \frac{1}{t}\right),$$

which provide equivalency of following transformation.

$$\frac{1}{2}\left(t^3 - \frac{1}{t^3} - 3\left(t - \frac{1}{t}\right)\right) + \frac{3}{2}\left(t - \frac{1}{t}\right) = d \iff \frac{1}{2}\left(t^3 - \frac{1}{t^3}\right) = d \iff$$

$$t^6 - 2dt^3 - 1 = 0 \iff \left[\begin{array}{l} t^3 = d - \sqrt{d^2+1} \\ t^3 = d + \sqrt{d^2+1} \end{array}\right. \iff \left[\begin{array}{l} t = \sqrt[3]{d - \sqrt{d^2+1}} \\ t = \sqrt[3]{d + \sqrt{d^2+1}} \end{array}\right.$$

and, finally gives us

(5).

$$z = \frac{\sqrt[3]{d + \sqrt{d^2+1}} - \sqrt[3]{\sqrt{d^2+1} - d}}{2};$$

2. $4z^3 - 3z = d$ and $|d| \le 1$.
Since $|d| \le 1$ then for any z which satisfy to equation $4z^3 - 3z = d$ holds $|z| \le 1$ (indeed, suppose that $|z| > 1$ then

$$1 \ge |d| = \left|4z^3 - 3z\right| \ge \left|4z^3\right| - |3z| = 4|z|^3 - 3|z| =$$

$$= |z|\left(4|z|^2 - 3\right) > |z|(4-3) = |z| > 1,$$

i.e. we get contradiction) and that gives us opportunity to apply substitution $z = \cos\varphi$, $\varphi \in [0, \pi]$. Denoting $\alpha := \cos^{-1}(d)$ we obtain

$$4z^3 - 3z = d \iff 4\cos^3\varphi - 3\cos\varphi = \cos\alpha \iff \cos 3\varphi = \cos\alpha \iff$$

$$\left\{\begin{array}{c} 3\varphi = \pm\alpha + 2k\pi \\ 0 \le \varphi \le \pi \end{array}\right. \iff \varphi \in \left\{\frac{\alpha}{3}, \frac{\alpha + 2\pi}{3}, \frac{2\pi - \alpha}{3}\right\}.$$

Thus we obtain

(6).

$$z_1 = \cos\frac{\alpha}{3}, z_2 = \cos\frac{\alpha + 2\pi}{3}, z_3 = \cos\frac{2\pi - \alpha}{3}.$$

In the case $|d| < 1$ roots z_1, z_2, z_3 are three different, because

$$0 < \frac{\alpha}{3} < \frac{\pi}{3}, \frac{2\pi}{3} < \frac{\alpha + 2\pi}{3} < \pi, \frac{\pi}{3} < \frac{2\pi - \alpha}{3} < \frac{2\pi}{3}.$$

If $d = 1$ then

$$4z^3 - 3z = 1 \iff 4z^3 - 3z - 1 = 0 \iff (2z+1)^2 (z-1) = 0$$

and, therefore, equation have one root $z = -\frac{1}{2}$ multiplicity 2 and one simple root $z = 1$.

From the other hand, since $\alpha = 0$ then

$$z_1 = \cos 0 = 1, z_2 = z_3 = \cos \frac{2\pi}{3} = -\frac{1}{2};$$

If $d = -1$ then

$$4z^3 - 3z = -1 \iff 4z^3 - 3z + 1 = 0 \iff (2z-1)^2 (z+1) = 0$$

and, therefore, equation have one root $z = \frac{1}{2}$ multiplicity 2 and one simple root $z = -1$.

From the other hand, since $\alpha = 0$ then

$$z_1 = \cos \pi = -1, z_2 = z_3 = \cos \frac{\pi}{3} = \frac{1}{2}.$$

So, if $|d| \le 1$ then equation $4z^3 - 3z = d$ have three real solution in the form

$$z_1 = \cos \frac{\alpha}{3}, z_2 = \cos \frac{\alpha + 2\pi}{3}, z_3 = \cos \frac{2\pi - \alpha}{3}.$$

3. $4z^3 - 3z = d$ and $|d| > 1$. Since $|d| > 1$ then for any z which satisfy to equation $4z^3 - 3z = d$ holds $|z| > 1$.

(Indeed, supposing the contrary $|z| \le 1$ we may use substitution $z = \cos\varphi$ and immediately obtain

$$1 < |d| = \left|4z^3 - 3z\right| = \left|4\cos^3\varphi - 3\cos\varphi\right| = |\cos 3\varphi| \le 1,$$

that is contradiction) and tis give us opportunity to use substitution $z = \frac{1}{2}\left(t + \frac{1}{t}\right)$ (because range of $t + \frac{1}{t}$ is set $(-\infty, 2] \cup [2, \infty)$).

Using this substitution we obtain equation

$$\frac{1}{2}\left(t + \frac{1}{t}\right)^3 - \frac{3}{2}\left(t + \frac{1}{t}\right) = d \iff \frac{1}{2}\left(t^3 + \frac{1}{t^3}\right) = d \iff$$

$$t^6 - 2dt^3 + 1 = 0 \iff \left[\begin{array}{l} t^3 = d - \sqrt{d^2 - 1} \\ t^3 = d + \sqrt{d^2 - 1} \end{array}\right. \iff \left[\begin{array}{l} t = \sqrt[3]{d - \sqrt{d^2 - 1}} \\ t = \sqrt[3]{d + \sqrt{d^2 - 1}} \end{array}\right.$$

and, finally gives us

(7).

$$z = \frac{\sqrt[3]{d + \sqrt{d^2 + 1}} + \sqrt[3]{d - \sqrt{d^2 + 1}}}{2}.$$

We will prove that equation $4z^3 - 3z = d$ have no other real solutions. Denote obtained solution via z_0 and supposing that there is another solution z distinct from z_0 we obtain

$$4z_1^3 - 3z_1 = 4z_0^3 - 3z_0 \iff$$

$$4z_1^2 + 4z_1 z_0 + 4z_0^2 - 3 = 0 \iff (2z_1 + z_0)^2 + 3\left(z_0^2 - 1\right) = 0 \implies z_0^2 \le 1,$$

that is contradiction to $|z_0| > 1$.

5. Criteria of the three real roots of cubic equation.

Lemma 4. Let k and l are real numbers, then system of equation

(8).

$$\left\{\begin{array}{c} x + y + z = 0 \\ xy + yz + zx = -k \\ xyz = l \end{array}\right|$$

have real solution iff $4k^3 \ge 27l^2$.

Proof. First consider case when $l = 0$.Then obvious that system solvable iff $k \ge 0$.
Thus, we can assume that $l > 0$, because solvability of the system (8) take place for l and $l := -l$ simultaneously.

1. *Necessity.* Suppose, that (x, y, z) is real solution of (8) (or, due to Vieta's Theorem, x, y, z are real solution of cubic equation $u^3 - ku - l = 0$). Since

$x+y+z=0$ and $x, y, z \neq 0$ then one of this three number is positive. Let it be z.
Then $x+y=-z$ and

$$xy=-k+z^2=\frac{l}{z} \iff z^3-kz-l=0.$$

Since $0 \leq (x-y)^2=(x+y)^2-4xy$ we obtain

$$0 \leq (-z)^2-4\left(-k+z^2\right) \iff 3z^2 \leq 4k$$

and

$$0 \leq (-z)^2-\frac{4l}{z} \iff 4l \leq z^3.$$

Thus for z we have equality $z^3-kz-l=0$ and and two inequalities: $z^3 \geq 4l$ and $3z^2 \leq 4k$,where from latter inequality follows $k \geq 0$.
Since $z^3 \geq 4l \iff z^6 \geq 16l^2$ and $3z^2 \leq 4k \iff 64k^3 \geq 27z^6$ then

$$16l^2 \leq z^6 \leq \frac{64k^3}{27} \iff \sqrt[3]{4l} \leq z \leq 2\sqrt{\frac{k}{3}},$$

which yield inequality

$$\sqrt[3]{4l} \leq 2\sqrt{\frac{k}{3}} \iff 27l^2 \leq 4k^3.$$

2. *Sufficiency.* Let $f(u) := u^3-ku-l$ and for coefficients k and l holds inequality $4k^3 \geq 27l^2$.
We have

$$f\left(\sqrt[3]{4l}\right)=4l-k\cdot\sqrt[3]{4l}-l=$$

$$=3l-k\sqrt[3]{4l} \leq 0 \iff 27l^3 \leq 4k^3 l \iff 27l^2 \leq 4k^3$$

$$f\left(2\sqrt{\frac{k}{3}}\right)=8\cdot\frac{k}{3}\sqrt{\frac{k}{3}}-2k\sqrt{\frac{k}{3}}-l=$$

$$=\frac{2}{3}k\sqrt{\frac{k}{3}}-l \geq 0 \iff 4k^3 \geq 27l^2.$$

From $f\left(\sqrt[3]{4l}\right) f\left(2\sqrt{\frac{k}{3}}\right) \leq 0$ follows that exist $z \in \left[\sqrt[3]{4l}, 2\sqrt{\frac{k}{3}}\right]$, such that $f(z)=0$.

But

$$\sqrt[3]{4l} \le z \iff 4l \le z^3$$

and

$$z \le 2\sqrt{\frac{k}{3}} \iff 4k - 3z^2 \ge 0$$

imply that exist real numbers x and y which simultaneously satisfy
$\begin{cases} x+y=-z \\ xy=\dfrac{l}{z} \end{cases}$ and $\begin{cases} x+y=-z \\ xy=-k+z^2 \end{cases}$
because

$$z^3 - kz - l = 0 \iff -k + z^2 = \frac{l}{z}.$$

Hence exist three real numbers x, y and z, for which
$x+y+z=0, xy+yz+zx=-k$ and $xyz=l$,or by the other words, cubic equation $u^3 - ku - l = 0$ have three real solution x, y and z.

Corollary. Cubic equation $u^3 + pu + q = 0$ have three real solution iff

(9).

$$27q^2 + 4p^3 \le 0.$$

Proof. Applying Lemma 4 to reduced cubic equation

$$u^3 + pu + q = 0$$

$(k := -p, q := -l)$ we obtain

$$27(-q)^2 \le 4(-p)^3 \iff 27q^2 + 4p^3 \le 0.$$

Theorem. Cubic equation $x^3 - ax^2 + bx - c = 0$ have three real solution (or correspondent Viet System $\left\{\begin{array}{c} x_1+x_2+x_3=a \\ x_1x_2+x_2x_3+x_3x_1=b \\ x_1x_2x_3=c \end{array}\right|$ solvable in real numbers) iff

(10).

$$\left(c-\frac{9ab-2a^3}{27}\right)^2 \leq$$

$$\leq \frac{4\left(a^2-3b\right)^3}{27^2} \iff b^2a^2-4b^3+18abc-4ca^3-27c^2 \geq 0.$$

Proof. Immediately follows by substitution in (9) translation formulas $q=\frac{ab}{3}-\frac{2a^3}{27}-c$ and $p=b-\frac{a^2}{3}$.

Remark. Since inequality **(10)** implies $a^2 \geq 3b$ then it can be equivalently rewritten as

$$\frac{9ab-2a^3-2\left(a^2-3b\right)\sqrt{a^2-3b}}{27} \leq$$

$$\leq q \leq \frac{9ab-2a^3+-2\left(a^2-3b\right)\sqrt{a^2-3b}}{27}$$

6. Maximization (minimization) of cubical polynomial-elementary approach. Suppose that cubical polynomial

$$F(x)=x^3-ax^2+bx-c$$

represented in the form

$$F(x)=(x-p)^2(x-q)+r$$

for some $p \neq q$ and r.

1. If $p>q$ then, for any $x \in [q,p]$, using AM-GM inequality we obtain

$$F(x)=r+\frac{(p-x)^2(2x-2q)}{2} \leq r+\frac{1}{2}\left(\frac{2(p-x)+(2x-2q)}{3}\right)^3=$$

$$=r+\frac{4(p-q)^3}{27},$$

where equality occurs iff

$$2x-2q=p-x \iff x=\frac{p+2q}{3}.$$

That is

$$\max_{x\in[q,p]} F(x) = F\left(\frac{p+2q}{3}\right) = r + \frac{4(p-q)^3}{27}.$$

2. If $p < q$ then for any $x \in [p,q]$ using AM-GM inequality we obtain

$$F(x) = r - \frac{(x-p)^2(2q-2x)}{2} \geq r - \frac{1}{2}\left(\frac{2(x-p)+(2q-2x)}{3}\right)^3 =$$

$$= r - \frac{4(q-p)^3}{27} = r + \frac{4(p-q)^3}{27},$$

where equality occurs iff

$$2q - 2x = x - p \iff x = \frac{p+2q}{3}.$$

That is

$$\min_{x\in[p,q]} F(x) = F\left(\frac{p+2q}{3}\right) = r + \frac{4(p-q)^3}{27}.$$

Thus, important to find condition which provide such representation.

Lemma 5. Cubical polynomial

$$F(x) = x^3 - ax^2 + bx - c$$

can be represented in form

$$F(x) = (x-p)^2(x-q) + r$$

if and only if equation $3x^2 - 2ax + b = 0$ have solution, i.e. $a^2 > 3b$.

Proof. Identity

$$(x-p)^2(x-q) + r = x^3 - x^2(2p+q) + x\left(p^2+2pq\right) + r - p^2q =$$

$$= x^3 - ax^2 + bx - c$$

yields

$$\begin{cases} 2p+q = a \\ p^2+2pq = b \\ p^2q - r = c \end{cases} \iff \begin{cases} q = a - 2p \\ r = p^2q - c \\ 3p^2 - 2pa + b = 0 \end{cases}$$

where equation

(D).

$$3p^2 - 2pa + b = 0$$

solvable iff $a^2 - 3b \geq 0$ but since in case $a^2 = 3b$ we obtain

$$x^3 - ax^2 + bx - c = x^3 - ax^2 + \frac{a^2}{3}x - c = \left(x - \frac{a}{3}\right)^3 + \frac{a^3}{27} - c,$$

i.e. $p = q = \dfrac{a}{3}$ then represented in form
$F(x) = (x-p)^2 (x-q) + r$ possible iff $a^2 > 3b$.
Since $a^2 > 3b$ yields two different solution p_1, p_2 of equation (D) then we obtain two representations

$$F(x) = (x - p_1)^2 (x - q_1) + r_1$$

and

$$F(x) = (x - p_2)^2 (x - q_2) + r_2.$$

Corollary. If $p_1 < p_2$ then

$$\max_{x \in (q_2, p_2)} F(x) = F(p_1)$$

and $\min\limits_{x \in (p_1, q_1)} F(x) = F(p_2)$.

Proof. Let $p_1 < p_2$. Since $q = a - 2p$ and $p_1 + p_2 = \dfrac{2a}{3}$ then

$$p_1 < \frac{p_1 + p_2}{2} < p_2 \iff p_1 < \frac{a}{3} < p_2 \iff q_1 = a - 2p_1 > p_1$$

and $q_2 = a - 2p_2 < p_2$.
Moreover, $\dfrac{p_1 + 2q_1}{3} = p_2 < q_1$ and $\dfrac{p_2 + 2q_2}{3} = p_1 > q_2$, i.e. $q_2 < p_1 < p_2 < q_1$.
Thus,

$$\max_{x \in [q_2, p_2]} F(x) = F(p_1)$$

and

$$\min_{x \in [p_1, q_1]} F(x) = F(p_2)$$

are local max and local min respectively.

7. Monotonicity of cubical polynomial without calculus.

Theorem.
$$P(x) = x^3 + ax^2 + bx + c$$
strictly increasing on $(-\infty, \infty)$ iff $a^2 \leq 3b$.

Proof.
1. Let $a^2 \leq 3b$. If $a^2 = 3b$ then

$$P(x) = \left(x - \frac{a}{3}\right)^3 + \frac{a^3}{27} - c$$

and, obviously, increasing on $(-\infty, \infty)$.
Consider now case $a^2 < 3b$.
Let $x_1 < x_2$. Then

$$P(x_2) - P(x_1) = x_2^3 - x_1^3 - a\left(x_2^2 - x_1^2\right) + b(x_2 - x_1) =$$

$$= (x_2 - x_1)\left(x_1^2 + x_1x_2 + x_2^2 - a(x_1 + x_2) + b\right) \geq$$

$$\geq (x_2 - x_1)\left(3\left(\frac{x_1 + x_2}{2}\right)^2 - 2a \cdot \frac{x_1 + x_2}{2} + b\right) > 0$$

because discriminant of $3x^2 - 2px + b$ is negative and

$$\frac{x_1^2 + x_1x_2 + x_2^2}{3} \geq \left(\frac{x_1 + x_2}{2}\right)^2 \iff$$

$$\iff 4x_1^2 + 4x_1x_2 + 4x_2^2 \geq 3\left(x_1^2 + 2x_1x_2 + x_2^2\right) \iff (x_1 - x_2)^2 \geq 0$$

for any x_1, x_2.

2. Let $P(x)$ strictly increasing on $(-\infty, \infty)$.
Since $P(x)$ strictly increasing on $(-\infty, \infty)$, then for any $x \in \mathbb{R}$ and any $h > 0$ holds

$$P(x + h) - P(x) >$$

$$> 0 \iff h\left(3x^2 + (3h+2a)\,x + \left(h^2 + ah + b\right)\right) > 0 \iff$$

$$\textbf{(11)}\quad 3x^2 + (3h+2a)\,x + \left(h^2 + ah + b\right) > 0.$$

Since inequality (11) holds for any real x then

$$D = (2a+3h)^2 - 12\left(h^2 + ah + b\right) < 0 \iff$$

$$4\left(a^2 - 3b\right) - 3h^2 < 0$$

and, therefore, $a^2 - 3b \leq 0$.
Indeed, if we assume that $a^2 - 3b > 0$ then $4\left(a^2 - 3b\right) - 3h^2 \geq 0$ for

$$h \geq 2\sqrt{\frac{a^2 - 3b}{3}}.$$

That is the contradiction because $4\left(a^2 - 3b\right) - 3h^2 < 0$ holds for any $h > 0$.
Remains consider behavior of

$$P(x) = x^3 + ax^2 + bx + c$$

if $a^2 > 3b$.

Lemma 6. Let $F(x) = x^3 + px + q$ then for any $x_1 < x_2$ there is point $x_0 \in (x_1, x_2)$ such that

$$\frac{F(x_2) - F(x_1)}{x_2 - x_1} = 3x_0^2 + p.$$

Proof. Since

$$\frac{F(x_2) - F(x_1)}{x_2 - x_1} = x_1^2 + x_1x_2 + x_2^2 + p$$

we have equation

$$3x_0^2 + p = x_1^2 + x_1x_2 + x_2^2 + p \iff 3x_0^2 =$$

$$= x_1^2 + x_1x_2 + x_2^2 \iff |x_0| = \sqrt{\frac{x_1^2 + x_1x_2 + x_2^2}{3}}.$$

If $x_1, x_2 > 0$ then

$$x_1 < \sqrt{\frac{x_1^2 + x_1x_2 + x_2^2}{3}} < x_2$$

and

$$x_0 = \sqrt{\frac{x_1^2 + x_1x_2 + x_2^2}{3}}$$

If $x_1, x_2 < 0$ then

$$x_1 < -\sqrt{\frac{x_1^2 + x_1x_2 + x_2^2}{3}} < x_2 \iff x_1^2 > \frac{x_1^2 + x_1x_2 + x_2^2}{3} > x_2^2$$

and

$$x_0 = -\sqrt{\frac{x_1^2 + x_1x_2 + x_2^2}{3}}$$

If $x_1 < 0 < x_2$ then holds at least one from two inequalities

$$0 < \sqrt{\frac{x_1^2 + x_1x_2 + x_2^2}{3}} < x_2$$

or

$$x_1 < -\sqrt{\frac{x_1^2 + x_1x_2 + x_2^2}{3}} < 0.$$

Indeed,

$$x_2^2 - \frac{x_1^2 + x_1x_2 + x_2^2}{3} = \frac{1}{3}(x_2 - x_1)(x_1 + 2x_2), x_1^2 - \frac{x_1^2 + x_1x_2 + x_2^2}{3} =$$

$$= -\frac{1}{3}(x_2 - x_1)(2x_1 + x_2)$$

and

$$\max\{x_1 + 2x_2, -(2x_1 + x_2)\} = \frac{x_2 - x_1 + 3|x_1 + x_2|}{2} > 0$$

and again x_0 is equal to one of two values $\sqrt{\frac{x_1^2 + x_1x_2 + x_2^2}{3}}$ or $-\sqrt{\frac{x_1^2 + x_1x_2 + x_2^2}{3}}$.

Theorem. (Mean Value (MV)Theorem for cubic without calculus). Let $P(x) = x^3 - ax^2 + bx - c$ then for any $x_1 < x_2$ there is point $x_0 \in (x_1, x_2)$ such that

$$\frac{P(x_2) - P(x_1)}{x_2 - x_1} = 3x_0^2 - 2ax_0 + b.$$

Proof. Let $t := x - \frac{a}{3}$ then applying Lemma to

$$F(t) := P\left(t + \frac{a}{3}\right) = t^3 + pu + q,$$

where

$$p = b - \frac{a^2}{3}, q = \frac{ab}{3} - \frac{2a^3}{27} - c$$

(Translation Formulas) and $t_i = x_i - \frac{a}{3}, i = 1, 2$ we obtain $t_0 \in (t_1, t_2)$ such that

$$\frac{P(x_2) - P(x_1)}{x_2 - x_1} = \frac{F(t_2) - F(t_1)}{t_2 - t_1} = 3t_0^2 + p$$

and since $x_0 = t_0 + \frac{a}{3}$ then

$$3t_0^2 + p = 3\left(x_0 - \frac{a}{3}\right)^2 + b - \frac{a^2}{3} = 3x_0^2 - 2ax_0 + b.$$

Corollary. If $a^2 > 3b$ then

$$P(x) = x^3 - ax^2 + bx - c$$

increase on $(-\infty, p_1]$ and on $[p_2, \infty)$, decrease on (p_1, p_2) (numbers $p_1 < p_2$ are solutions of equation (D)).

Proof. Since $3x^2 - 2xa + b < 0$ for $x \in (p_1, p_2)$ then for any $p_1 \leq x_1 < x_2 \leq p_2$ by MV Theorem there is $x_0 \in (x_1, x_2)$ such that

$$\frac{P(x_2) - P(x_1)}{x_2 - x_1} = 3x_0^2 - 2ax_0 + b < 0$$

and, therefore, $P(x)$ decrease on (p_1, p_2).
Similarly, for $x_1 < x_2 \leq p_1$ (or $p_2 \leq x_1 < x_2$) there is $x_0 \in (x_1, x_2)$ such that

$$\frac{P(x_2) - P(x_1)}{x_2 - x_1} = 3x_0^2 - 2ax_0 + b > 0$$

because $3x^2 - 2xa + b > 0$ for $x \in (-\infty, p_1] \cup [p_2, \infty)$.
Hence, $P(x)$ increase on $(-\infty, p_1]$ and on $[p_2, \infty)$.

8. Exploring convexity. First note that $Q(x) := x^3 + px + q$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.
Indeed, since linear function $l(x) = px + q$ simulteneously concave ap and concave down on $(-\infty, \infty)$ and for any $x_1, x_2 > 0$ holds

$$\frac{x_1^3 + x_2^3}{2} \geq \left(\frac{x_1 + x_2}{2}\right)^3$$

(because $4\left(x_1^2 - x_1x_2 + x_2^2\right) - (x_1 + x_2)^2 = 3(x_1 - x_2)^2 \geq 0$) then

$$\frac{Q(x_1) + Q(x_2)}{2} \geq Q\left(\frac{x_1 + x_2}{2}\right),$$

that is $Q(x)$ is concave up on $(0, \infty)$.
If $x_1, x_2 < 0$ then

$$\frac{(-x_1)^3 + (-x_2)^3}{2} \geq \left(\frac{(-x_1) + (-x_2)}{2}\right)^3 \iff \frac{x_1^3 + x_2^3}{2} \leq \left(\frac{x_1 + x_2}{2}\right)^3$$

and, therefore,

$$\frac{Q(x_1) + Q(x_2)}{2} \leq Q\left(\frac{x_1 + x_2}{2}\right),$$

that is $Q(x)$ is concave down on $(-\infty, 0)$.
Let $F(x) = x^3 - ax^2 + bx - c$. Since

$$Q(t) = F\left(t - \frac{a}{3}\right) = t^3 + pu + q,$$

where $p = \dfrac{3b - a^2}{3}, q = \dfrac{2a^3 - 9ba + 27c}{27}$ (Translation Formulas) we can immediatelly conclude that $F(x)$ is is concave up on $\left(\dfrac{a}{3}, \infty\right)$ and concave down on $\left(-\infty, \dfrac{a}{3}\right)$.

8. Problems.

Problem 1.(1983 Bulgarian Winter Competition). Determine all values of parameter p for which system of equations

$$\textbf{(O)} \qquad \left\{ \begin{array}{c} x+y+z=2 \\ xy+yz+zx=1 \\ xyz=p \end{array} \right|$$

have real solutions.

Solution 1. Original problem have another equivalent setting.
Determine all values of parameter p for which equation

$$u^3-2u^3+u-p=0$$

have three real solutions.
By substitution $u=v+\frac{2}{3}$ we get the same question to reduced cubic equation

$$v^3-\frac{v}{3}+\frac{2}{27}-p=0.$$

This equation have three real roots iff

$$27\left(\frac{2}{27}-p\right)^2+4\left(-\frac{1}{3}\right)^3\leq 0 \iff \left(\frac{2}{27}-p\right)^2\leq$$

$$\leq\frac{4}{27^2} \iff \left|p-\frac{2}{27}\right|\leq\frac{2}{27} \iff$$

$$0\leq p\leq\frac{4}{27}.$$

Solution 2. Let $x=a+\frac{2}{3}, y=b+\frac{2}{3}, z=c+\frac{2}{3}$.Then

$$a+b+c=0, \sum_{cic} xy=$$

$$=\sum_{cic}\left(a+\frac{2}{3}\right)\left(b+\frac{2}{3}\right)=ab+bc+ca+\frac{4}{3}=1 \iff ab+bc+ca=-\frac{1}{3},$$

$$abc=\left(x-\frac{2}{3}\right)\left(y-\frac{2}{3}\right)\left(z-\frac{2}{3}\right)=$$

$$=xyz-\frac{2\left(xy+yz+zx\right)}{3}+\frac{4}{9}\left(x+y+z\right)-\frac{8}{27}=$$

$$= p - \frac{2}{3} + \frac{8}{9} - \frac{8}{27} = p - \frac{2}{27}.$$

Thus we obtain the system

$$\textbf{(E)} \ \$ \quad \left\{ \begin{array}{c} a+b+c=0 \\ ab+bc+ca=-\dfrac{1}{3} \\ abc=q \end{array} \right| ,$$

where $q = p - \dfrac{2}{27}$ and which equivalent to the original system.
Note that system **(E)** solvable iff solvable the system

$$\textbf{(E1)} \quad \left\{ \begin{array}{c} a+b+c=0 \\ ab+bc+ca=-\dfrac{1}{3} \\ abc=|q| \end{array} \right| .$$

Also note that at least one of three numbers a, b, c must be non-negative, because otherwise $ab + bc + ca > 0$. Let it be c.
Since $a + b = -c$ and $ab = c^2 - \dfrac{1}{3} = |q|$ then we should claim

$$c^2 - 4\left(c^2 - \frac{1}{3}\right) \geq 0 \iff c^2 \leq \frac{4}{9} \iff c \leq \frac{2}{3}$$

and

$$c^2 - \frac{4}{c}|q| \geq 0 \iff c^3 \geq 4|q| \iff c \geq \sqrt[3]{4|q|}.$$

Hence,

$$\frac{2}{3} \geq \sqrt[3]{4|q|} \iff \frac{8}{27} \geq$$

$$\geq 4|q| \iff |q| \leq \frac{2}{27} \iff -\frac{2}{27} \leq p - \frac{2}{27} \leq \frac{2}{27} \iff$$

$$0 \leq p \leq \frac{4}{27}.$$

Solution 3. First we will find range of z if $x + y + z = 2$ and $xy + yz + zx = 1$. Since $x + y = 2 - z$ and

$$xy + yz + zx = 1 \iff xy = 1 - z(2 - z) \iff xy = (z-1)^2$$

then x, y should be solutions of the Vieta's System $\begin{cases} x+y=2-z \\ (z-1)^2 \end{cases}$ which is solvable iff

$$(2-z)^2 - 4(z-1)^2 \geq 0 \iff z(3z-4) \leq 0 \iff 0 \leq z \leq \frac{4}{3}.$$

Thus range of p is $\left[\min_z p, \max_z p\right]$, where

$$p = z^3 - 2z^2 + z = z(z-1)^2.$$

Obvious that $\min_{z\in[0,4/3]} p = 0$. Since for $0 \leq z \leq 1$ by AM-GM Inequality

$$z(z-1)^2 = \frac{1}{2} \cdot 2z(1-z)^2 \leq \frac{1}{2}\left(\frac{2z+2-2z}{3}\right)^3 = \frac{4}{27}$$

with equality condition

$$2z = 1 - z \iff z = \frac{1}{3}.$$

Hence, $\max_{z\in[0,1]} p = \frac{4}{27}$.Since $z(z-1)^2$ is increasing for $z \geq 1$ then

$$\max_{z\in[1,4/3]} p = \frac{4}{3}\left(\frac{4}{3}-1\right)^2 = \frac{4}{27}$$

and, therefore,

$$\max_{z\in[0,4/3]} p = \frac{4}{27}.$$

Thus, $range\,(p) = \left[0, \frac{4}{27}\right]$.

Problem 2.(Vietnamese Math.Olymp.1999,Category B, Problem 3). Consider real numbers a, b such that all roots of equation

$$ax^3 - x^2 + bx - 1 = 0$$

are real and positive. Determine the smallest possible value of the following expression:

$$P(a,b) = \frac{5a^2 - 3ab + 2}{a^2(b-a)}.$$

Solution. Let x_1, x_2, x_3 be the positive roots of equation

$$ax^3 - x^2 + bx - 1 = 0.$$

Then

$$\begin{cases} x_1 + x_2 + x_3 = \dfrac{1}{a} \\ x_1x_2 + x_2x_3 + x_3x_1 = \dfrac{b}{a} \\ x_1x_2x_3 = \dfrac{1}{a} \end{cases}$$

and using AM-GM Inequality we obtain

$$\frac{1}{a} = x_1x_2x_3 \le \left(\frac{x_1 + x_2 + x_3}{3}\right)^3 = \frac{1}{27a^3} \iff a^2 \le \frac{1}{27} \iff a \le \frac{1}{3\sqrt{3}}.$$

Also we have

$$\frac{b}{a} = x_1x_2 + x_2x_3 + x_3x_1 \le \frac{(x_1 + x_2 + x_3)^2}{3} = \frac{1}{3a^2} \iff 3ab \le 1.$$

Substitution $x = \dfrac{1}{t}$ in equation $ax^3 - x^2 + bx - 1 = 0$ give us monic cubic equation $t^3 - bt^2 + t - a = 0$ which have three positive real roots $t_i = \dfrac{1}{x_i}, i = 1, 2, 3$. Then for t_1, t_2, t_3 we have

$$\begin{cases} t_1 + t_2 + t_3 = b \\ t_1t_2 + t_2t_3 + t_3t_1 = 1 \\ t_1t_2t_3 = a \end{cases}.$$

Hence,

$$\frac{b^2}{3} = \frac{(t_1 + t_2 + t_3)^2}{3} \ge t_1t_2 + t_2t_3 + t_3t_1 = 1 \implies b \ge \sqrt{3}.$$

Since $b \ge \sqrt{3}$, $a \le \dfrac{1}{3\sqrt{3}}$ and $3ab \le 1 \iff ab \le \dfrac{1}{3}$, then

$$b - a \ge \sqrt{3} - \frac{1}{3\sqrt{3}} = \frac{8}{3\sqrt{3}} > 0,\ 5a^2 - 3ab + 2 \ge 5a^2 + 1$$

and

$$a^2(b - a) = a\left(ab - a^2\right) \le a\left(\frac{1}{3} - a^2\right) = \frac{a\left(1 - 3a^2\right)}{3}.$$

Hence,

$$P(a,b) = \left(5a^2 - 3ab + 2\right) \cdot \frac{1}{a(ab - a^2)} \geq \frac{3\left(5a^2+1\right)}{a(1-3a^2)}.$$

Let $h(a) = \dfrac{5a^2+1}{a(1-3a^2)}$. We will prove that $h(a) \geq h\left(\dfrac{1}{3\sqrt{3}}\right)$ for $0 < a \leq \dfrac{1}{3\sqrt{3}}$.

We have

$$h(a) \geq h\left(\frac{1}{3\sqrt{3}}\right) \iff \frac{5a^2+1}{a(1-3a^2)} \geq \frac{\frac{5}{27}+1}{\frac{1}{3\sqrt{3}}\left(1-\frac{1}{9}\right)} = 4\sqrt{3} \iff$$

$$\iff 5a^2 + 1 \geq 4\sqrt{3}a - 12\sqrt{3}a^3 \iff 12\sqrt{3}a^3 + 5a^2 - 4\sqrt{3}a + 1 \geq 0.$$

Let $c := \sqrt{3}a$ then $c \in \left(0, \dfrac{1}{3}\right]$ and

$$12\sqrt{3}a^3 + 5a^2 - 4\sqrt{3}a + 1 =$$

$$= 4c^3 + \frac{5}{3}c^2 - 4c + 1 = \frac{1}{3}\left(12c^3 + 5c^2 - 12c + 3\right) =$$

$$= \frac{1}{3}(3c-1)\left(4c^2 + 3c - 3\right) \geq 0$$

because $3c - 1 \leq 0$ and $4c^2 + 3c - 3 \leq \dfrac{4}{9} + 1 - 3 < 0$ for $c \in \left(0, \dfrac{1}{3}\right]$.

Since $P(a,b) \geq 12\sqrt{3} = P\left(\dfrac{1}{3\sqrt{3}}, \sqrt{3}\right)$ then $\min P(a,b) = 12\sqrt{3}$.

Substitution $a = \dfrac{1}{3\sqrt{3}}$ and $b = \sqrt{3}$ in equation $ax^3 - x^2 + bx - 1 = 0$ gives

$$\frac{1}{3\sqrt{3}}x^3 - x^2 + \sqrt{3}x - 1 = 0 \iff \left(x - \sqrt{3}\right)^3 = 0.$$

As a complement:

Problem 1.

a). Prove that $\sqrt[3]{9+\sqrt{80}} + \sqrt[3]{9-\sqrt{80}} = 3$;

b). Prove that $\sqrt[3]{54+30\sqrt{3}} + \sqrt[3]{54-30\sqrt{3}} = 6$;

c). Prove that $\dfrac{\sqrt[3]{45+29\sqrt{2}}+\sqrt[3]{45-29\sqrt{2}}}{2}$ is integer number.

Problem 2. Calculate following sum if known that x_1, x_2, x_3 are roots of cubic equation $F(x)=0$:

a). $\dfrac{1}{2-x_1}+\dfrac{1}{2-x_2}+\dfrac{1}{2-x_3}$, $F(x)=x^3-3x-1=0$;

b). $\dfrac{1}{x_1^2-3x_1+2}+\dfrac{1}{x_2^2-3x_2+2}+\dfrac{1}{x_3^2-3x_3+2}$, $F(x)=x^3+x^2-4x+1$;

c). $\dfrac{1}{x_1^2-2x_1+1}+\dfrac{1}{x_2^2-2x_2+1}+\dfrac{1}{x_3^2-2x_3+1}$, $F(x)=x^3+x^2-1$.

Problem 3. For any cubic polynomial $f(x)=x^3+ax^2+bx+c$ there is natural n such that $\sum\limits_{k=0}^{n} f(x+k)>0$ have only one real solution in $\mathbb{R}$.
Hint:v Suffice to prove that for any quadratic trinomial x^2+px+q there is natural n such that $\sum\limits_{k=0}^{n}\left((x+k)^2+p(x+k)+q\right)>0$ for any real x.

Problem 4. For real x, y, z find the *range* (xyz) if $x+y+z=5, xy+yz+zx=8$.

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